VECTORS

I. **Position Vector Of A Point**:

let O be a fixed origin, then the position vector of a point P is the vector $\overrightarrow{OP}$. If $\vec{a}$ and $\vec{b}$ are position vectors of two points A and B, then,

$\overrightarrow{AB} = \vec{b} - \vec{a}$ = pv of B − pv of A.

DISTANCE FORMULA : Distance between the two points A $(\vec{a})$ and B $(\vec{b})$ is AB = $\left|\vec{a} - \vec{b}\right|$

SECTION FORMULA : $\vec{r} = \dfrac{n\vec{a} + m\vec{b}}{m + n}$. Mid point of AB = $\dfrac{\vec{a} + \vec{b}}{2}$.

II. **Scalar Product Of Two Vectors:** $\vec{a}.\vec{b} = |\vec{a}||\vec{b}|\cos\theta$, where $|\vec{a}|, |\vec{b}|$ are magnitude of $\vec{a}$ and $\vec{b}$ respectively and θ is angle between $\vec{a}$ and $\vec{b}$.

1. i.i = j.j = k.k = 1; i.j = j.k = k.i = 0 ☞ projection of $\vec{a}$ on $\vec{b} = \dfrac{\vec{a}.\vec{b}}{|\vec{b}|}$

2. If $\vec{a} = a_1 i + a_2 j + a_3 k$ & $\vec{b} = b_1 i + b_2 j + b_3 k$ then $\vec{a}.\vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$

3. The angle ϕ between $\vec{a}$&$\vec{b}$ is given by $\cos\phi = \dfrac{\vec{a}.\vec{b}}{|\vec{a}|\,|\vec{b}|}$, $0 \le \phi \le \pi$

4. $\vec{a}.\vec{b} = 0 \Leftrightarrow \vec{a} \perp \vec{b}$ $(\vec{a} \neq 0\ \vec{b} \neq 0)$

III. **Vector Product Of Two Vectors:**

1. If $\vec{a}$&$\vec{b}$ are two vectors & θ is the angle between them then $\vec{a}\text{x}\vec{b} = |\vec{a}||\vec{b}|\sin\theta\vec{n}$, where $\vec{n}$ is the unit vector perpendicular to both $\vec{a}$&$\vec{b}$ such that $\vec{a}, \vec{b}$&$\vec{n}$ forms a right handed screw system.

2. Geometrically $|\vec{a}\text{x}\vec{b}|$ = area of the parallelogram whose two adjacent sides are represented by $\vec{a}$&$\vec{b}$.

3. $\hat{i}\times\hat{i} = \hat{j}\times\hat{j} = \hat{k}\times\hat{k} = \vec{0}$; $\hat{i}\times\hat{j} = \hat{k}, \hat{j}\times\hat{k} = \hat{i}, \hat{k}\times\hat{i} = \hat{j}$

4. If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ & $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ then

$$\vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

5. $\vec{a}\times\vec{b} = \vec{o} \Leftrightarrow \vec{a}$ and $\vec{b}$ are parallel (collinear)

$(\vec{a} \neq 0, \vec{b} \neq 0)$ i.e. $\vec{a} = K\vec{b}$, where K is a scalar.

6. Unit vector perpendicular to the plane of $\vec{a}$ & $\vec{b}$ is $\hat{n} = \pm \dfrac{\vec{a} \times \vec{b}}{|\vec{a} \times \vec{b}|}$

☞ If $\vec{a}, \vec{b} \& \vec{c}$ are the pv's of 3 points A, B & C then the vector area of triangle ABC = $\frac{1}{2}\left[\vec{a}\times\vec{b} + \vec{b}\times\vec{c} + \vec{c}\times\vec{a}\right]$.

The points A, B & C are collinear if $\vec{a} \times \vec{b} + \vec{b} \times \vec{c} + \vec{c} \times \vec{a} = \vec{0}$

☞ Area of any quadrilateral whose diagonal vectors are $\vec{d}_1$ & $\vec{d}_2$ is given by $\frac{1}{2}\left|\vec{d}_1 \times \vec{d}_2\right|$

☞ Lagrange's Identity : $(\vec{a}\times\vec{b})^2 = |\vec{a}|^2|\vec{b}|^2 - (\vec{a}.\vec{b})^2 = \begin{vmatrix} \vec{a}.\vec{a} & \vec{a}.\vec{b} \\ \vec{a}.\vec{b} & \vec{b}.\vec{b} \end{vmatrix}$

IV. Scalar Triple Product:

☞ The scalar triple product of three vectors $\vec{a}$, $\vec{b}$ & $\vec{c}$ is defined as:

$$\vec{a} \times \vec{b} . \vec{c} = |\vec{a}||\vec{b}||\vec{c}| \sin\theta \cos\phi .$$

☞ Volume of tetrahydron $V = [\vec{a}\vec{b}\vec{c}]$

☞ In a scalar triple product the position of dot & cross can be interchanged i.e.

$$\vec{a} . (\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}) . \vec{c} \quad \text{OR} \quad [\vec{a}\ \vec{b}\ \vec{c}] = [\vec{b}\ \vec{c}\ \vec{a}] = [\vec{c}\ \vec{a}\ \vec{b}]$$

☞ $\vec{a} . (\vec{b} \times \vec{c}) = -\vec{a} . (\vec{c} \times \vec{b})$ i.e. $[\vec{a}\ \vec{b}\ \vec{c}] = -[\vec{a}\ \vec{c}\ \vec{b}]$

☞ If $\vec{a} = a_1i + a_2j + a_3k$; $\vec{b} = b_1i + b_2j + b_3k$ & $\vec{c} = c_1i + c_2j + c_3k$ then $[\vec{a}\vec{b}\vec{c}] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$.

☞ If $\vec{a}$, $\vec{b}$, $\vec{c}$ are coplanar $\Leftrightarrow [\vec{a}\vec{b}\vec{c}] = 0$.

☞ Volume of tetrahedron OABC with O as origin & A($\vec{a}$), B($\vec{b}$) and C($\vec{c}$) be the vertices = $\left| \frac{1}{6}[\vec{a}\ \vec{b}\ \vec{c}] \right|$

☞ The positon vector of the centroid of a tetrahedron if the pv's of its vertices are $\vec{a}$, $\vec{b}$, $\vec{c}$ & $\vec{d}$ are given by $\frac{1}{4}$ $[\vec{a} + \vec{b} + \vec{c} + \vec{d}]$.

V. Vector Triple Product:

$\vec{a}$ x $(\vec{b}$ x $\vec{c})$ = $(\vec{a} . \vec{c})\vec{b} - (\vec{a} . \vec{b})\vec{c}$, $(\vec{a}$ x $\vec{b})$ x $\vec{c}$ = $(\vec{a} . \vec{c})\vec{b} - (\vec{b} . \vec{c})\vec{a}$

☞ $(\vec{a}$ x $\vec{b})$ x $\vec{c} \neq \vec{a}$ x $(\vec{b}$ x $\vec{c})$, in general

3-DIMENSION

1. Vector representation of a point :

Position vector of point P (x, y, z) is $x\hat{i} + y\hat{j} + z\hat{k}$.

2. Distance formula :

$\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$, AB = $|\overrightarrow{OB} - \overrightarrow{OA}|$

3. Distance of P from coordinate axes :

PA = $\sqrt{y^2 + z^2}$, PB = $\sqrt{z^2 + x^2}$, PC = $\sqrt{x^2 + y^2}$

4. Section Formula : $x = \frac{mx_2 + nx_1}{m+n}$, $y = \frac{my_2 + ny_1}{m+n}$, $z = \frac{mz_2 + nz_1}{m+n}$

Mid point : $x = \frac{x_1 + x_2}{2}$, $y = \frac{y_1 + y_2}{2}$, $z = \frac{z_1 + z_2}{2}$

5. Direction Cosines And Direction Ratios

(i) Direction cosines**:** Let α, β, γ be the angles which a directed line makes with the positive directions of the axes of x, y and z respectively, then cos α, cosβ, cos γ are called the direction cosines of the line. The direction cosines are usually denoted by (ℓ, m, n). Thus ℓ = cos α, m = cos β, n = cos γ.

(ii) If ℓ, m, n be the direction cosines of a line, then $\ell^2 + m^2 + n^2 = 1$

(iii) Direction ratios: Let a, b, c be proportional to the direction cosines ℓ, m, n then a, b, c are called the direction ratios.

(iv) If ℓ, m, n be the direction cosines and a, b, c be the direction ratios of a vector, then

$$\ell = \pm\frac{a}{\sqrt{a^2 + b^2 + c^2}}, m = \pm\frac{b}{\sqrt{a^2 + b^2 + c^2}}, n = \pm\frac{c}{\sqrt{a^2 + b^2 + c^2}}$$

(vi) If the coordinates P and Q are (x_1, y_1, z_1) and (x_2, y_2, z_2) then the direction ratios of line PQ are, $a = x_2 - x_1$, $b = y_2 - y_1$ & $c = z_2 - z_1$ and

the direction cosines of line PQ are $\ell = \frac{x_2 - x_1}{|PQ|}$,

$$m = \frac{y_2 - y_1}{|PQ|} \text{ and } n = \frac{z_2 - z_1}{|PQ|}$$

6. Angle Between Two Line Segments:

$$\cos\theta = \left| \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}} \right|.$$

The line will be perpendicular if $a_1a_2 + b_1b_2 + c_1c_2 = 0$,

parallel if $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$

7. Projection of a line segment on a line

If $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ then the projection of PQ on a line having direction cosines ℓ, m, n is

$$\left| \ell(x_2 - x_1) + m(y_2 - y_1) + n(z_2 - z_1) \right|$$

8. Equation Of A Plane :

General form: $ax + by + cz + d = 0$, where a, b, c are not all zero, $a, b, c, d \in R$.

(i) Normal form : $\ell x + my + nz = p$

(ii) Plane through the point (x_1, y_1, z_1) :
$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$

(iii) Intercept Form: $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$

(iv) Vector form: $(\vec{r} - \vec{a}).\vec{n} = 0$ or $\vec{r}.\vec{n} = \vec{a}.\vec{n}$

(v) Any plane parallel to the given plane $ax + by + cz + d = 0$ is $ax + by + cz + \lambda = 0$. Distance between $ax + by + cz + d_1 = 0$ and $ax + by + cz + d_2 = 0$ is $= \frac{|d_1 - d_2|}{\sqrt{a^2 + b^2 + c^2}}$

(vi) **Equation of a plane passing through a given point & parallel to the given vectors:**

$\vec{r} = \vec{a} + \lambda\vec{b} + \mu\vec{c}$ (parametric form) where λ & μ are scalars.

or $\vec{r}\cdot(\vec{b}\times\vec{c}) = \vec{a}\cdot(\vec{b}\times\vec{c})$ (non parametric form)

9. A Plane & A Point

(i) Distance of the point (x', y', z') from the plane $ax + by + cz + d = 0$ is given by $\dfrac{ax'+by'+cz'+d}{\sqrt{a^2+b^2+c^2}}$.

(ii) Length of the perpendicular from a point ($\vec{a}$) to plane $\vec{r}\cdot\vec{n} = d$ is given by $p = \dfrac{|\vec{a}\cdot\vec{n} - d|}{|\vec{n}|}$.

(iii) Foot (x', y', z') of perpendicular drawn from the point (x_1, y_1, z_1) to the plane $ax + by + cz + d = 0$ is given by $\dfrac{x'-x_1}{a} = \dfrac{y'-y_1}{b} = \dfrac{z'-z_1}{c}$

$$= -\frac{(ax_1+by_1+cz_1+d)}{a^2+b^2+c^2}$$

(iv) **To find image of a point w.r.t. a plane:**
Let P (x_1, y_1, z_1) is a given point and $ax + by + cz + d = 0$ is given plane Let (x', y', z') is the image point. then

$$\frac{x'-x_1}{a} = \frac{y'-y_1}{b} = \frac{z'-z_1}{c} = -2\,\frac{(ax_1+by_1+cz_1+d)}{a^2+b^2+c^2}$$

10. Angle Between Two Planes:

$$\cos\theta = \left|\frac{aa'+bb'+cc'}{\sqrt{a^2+b^2+c^2}\,\sqrt{a'^2+b'^2+c'^2}}\right|$$

Planes are perpendicular if $aa' + bb' + cc' = 0$ and planes are parallel if

$$\frac{a}{a'} = \frac{b}{b'} = \frac{c}{c'}$$

The angle θ between the planes $\vec{r}\,.\,\vec{n}_1 = d_1$ and $\vec{r}\,.\,\vec{n}_2 = d_2$ is given by, cos

$$\theta = \frac{\vec{n}_1\,.\,\vec{n}_2}{|\vec{n}_1|\,.\,|\vec{n}_2|}$$

Planes are perpendicular if $\vec{n}_1\,.\,\vec{n}_2 = 0$ & planes are parallel if

$\vec{n}_1 = \lambda\,\vec{n}_2$, λ is a scalar

11. Angle Bisectors

(i) The equations of the planes bisecting the angle between two given planes
$a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$ are

$$\frac{a_1x + b_1y + c_1z + d_1}{\sqrt{a_1^2 + b_1^2 + c_1^2}} = \pm\, \frac{a_2x + b_2y + c_2z + d_2}{\sqrt{a_2^2 + b_2^2 + c_2^2}}$$

(ii) Bisector of acute/obtuse angle: First make both the constant terms positive. Then
$a_1a_2 + b_1b_2 + c_1c_2 > 0 \quad \Rightarrow$ origin lies on obtuse angle
$a_1a_2 + b_1b_2 + c_1c_2 < 0 \quad \Rightarrow$ origin lies in acute angle

12. Family of Planes

(i) Any plane through the intersection of $a_1x + b_1y + c_1z + d_1 = 0$ & $a_2x + b_2y + c_2z + d_2 = 0$ is
$a_1x + b_1y + c_1z + d_1 + \lambda\,(a_2x + b_2y + c_2z + d_2) = 0$

(ii) The equation of plane passing through the intersection of the planes $\vec{r}\,.\,\vec{n}_1 = d_1$ & $\vec{r}\,.\,\vec{n}_2 = d_2$ is $\vec{r}\,.\,(n_1 + \lambda\,\vec{n}_2) = d_1 + \lambda d_2$ where λ is arbitrary scalar

13. Volume Of A Tetrahedron: Volume of a tetrahedron with vertices A (x_1, y_1, z_1), B(x_2, y_2, z_2), C (x_3, y_3, z_3) and

D (x_4, y_4, z_4) is given by $V = \frac{1}{6}\begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix}$

A LINE

1. Equation Of A Line

(i) A straight line is intersection of two planes.
it is reprsented by two planes $a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + +c_2z + d_2 = 0$.

(ii) Symmetric form : $\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c} = r$.

(iii) Vector equation: $\vec{r} = \vec{a} + \lambda \vec{b}$

(iv) Reduction of cartesion form of equation of a line to vector form & vice versa

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c} \Leftrightarrow \vec{r} = (x_1\hat{i} + y_1\hat{j} + z_1\hat{k}) + \lambda(a\hat{i} + b\hat{j} + c\hat{k}).$$

2. Angle Between A Plane And A Line:

(i) If θ is the angle between line $\frac{x - x_1}{\ell} = \frac{y - y_1}{m} = \frac{z - z_1}{n}$ and the plane $ax + by + cz + d = 0$, then

$$\sin\theta = \left| \frac{a\ell + bm + cn}{\sqrt{(a^2 + b^2 + c^2)}\sqrt{\ell^2 + m^2 + n^2}} \right|.$$

(ii) Vector form: If θ is the angle between a line $\vec{r} = (\vec{a} + \lambda\vec{b})$ and $\vec{r} \cdot \vec{n} = d$ then $\sin\theta = \left[\frac{\vec{b}.\vec{n}}{|\vec{b}||\vec{n}|} \right]$.

(iii) Condition for perpendicularity $\frac{\ell}{a} = \frac{m}{b} = \frac{n}{c}$, $\quad \vec{b} \times \vec{n} = 0$

(iv) Condition for parallel $\quad a\ell + bm + cn = 0 \quad \vec{b}.\vec{n} = 0$

3. Condition For A Line To Lie In A Plane

(i) Cartesian form: Line $\frac{x - x_1}{\ell} = \frac{y - y_1}{m} = \frac{z - z_1}{n}$ would lie in a plane $ax + by + cz + d = 0$, if $ax_1 + by_1 + cz_1 + d = 0$ & $a\ell + bm + cn = 0$.

(ii) Vector form: Line $\vec{r} = \vec{a} + \lambda\vec{b}$ would lie in the plane $\vec{r}.\vec{n} = d$ if $\vec{b}.\vec{n} = 0$ & $\vec{a}.\vec{n} = d$

4. Skew Lines:

(i) The straight lines which are not parallel and non–coplanar i.e. non–intersecting are called skew lines.

lines $\frac{x-\alpha}{\ell} = \frac{y-\beta}{m} = \frac{z-\gamma}{n}$ & $\frac{x-\alpha'}{\ell'} = \frac{y-\beta'}{m'} = \frac{z-\gamma'}{n'}$

If $\Delta = \begin{vmatrix} \alpha'-\alpha & \beta'-\beta & \gamma'-\gamma \\ \ell & m & n \\ \ell' & m' & n' \end{vmatrix} \neq 0$, then lines are skew.

(ii) Shortest distance formula for lines

$\vec{r} = \vec{a}_1 + \lambda \vec{b}_1$ and $\vec{r} = \vec{a}_2 + \lambda \vec{b}_2$ is $d = \left| \frac{(\vec{a}_2 - \vec{a}_1).(\vec{b}_1 \times \vec{b}_2)}{|\vec{b}_1 \times \vec{b}_2|} \right|$

(iii) Vector Form: For lines $\vec{r} = \vec{a}_1 + \lambda \vec{b}_1$ and $\vec{r} = \vec{a}_2 + \lambda \vec{b}_2$ to be skew

$(\vec{b}_1 \times \vec{b}_2).(\vec{a}_2 - \vec{a}_1) \neq 0$

(iv) Shortest distance between parallel lines

$\vec{r} = \vec{a}_1 + \lambda \vec{b}$ & $\vec{r} = \vec{a}_2 + \mu \vec{b}$ is $d = \left| \frac{(\vec{a}_2 - \vec{a}_1) \times \vec{b}}{|\vec{b}|} \right|$.

(v) Condition of coplanarity of two lines $\vec{r} = \vec{a} + \lambda \vec{b}$ & $\vec{r} = \vec{c} + \mu \vec{d}$ is

$[\vec{a}-\vec{c} \;\; \vec{b} \;\; \vec{d}] = 0$

5. Sphere

General equation of a sphere is $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$. $(-u, -v, -w)$ is the centre and $\sqrt{u^2 + v^2 + w^2 - d}$ is the radius of the sphere.